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JANUARY EXAMINATIONS 2009

Bachelor of Science: Year 3
Master of Physics: Year 3
Master of Physics: Year 4

STATISTICAL AND LOW TEMPERATURE PHYSICS

TIME ALLOWED: THREE HOURS

INSTRUCTIONS TO CANDIDATES

Answer **all** questions.

Question 1 carries 50% of the total marks.

Questions 2 and 3 each carry 25% of the total marks.

The marks allotted to each part of a question are shown in square brackets.

In the event of a student answering both parts of an either/or question and not clearly crossing out one answer, only the answer to part (a) of the question will be marked.

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1. Answer **all** parts.

a) A set of 6 distinguishable particles can occupy energy states 0, ε, 2ε, 3ε and 4ε. The total energy of the set is 4ε.

(i) Write out the five possible distributions of the particles in the energy states.

(ii) Giving the appropriate formula, or explaining your method, find the number of microstates for each distribution.

(iii) Evaluate the mean population for each energy state. Give the formula or explain the method that you use.

(iv) If, instead of being distinguishable, the particles had been indistinguishable bosons, evaluate the mean population of each energy state.

[8]

b) N atoms bound in a solid system can exist in levels of energy ε and 2ε. The level of energy ε contains two states (has a degeneracy of two), while the level of energy 2ε contains a single state.

(i) Write an expression for the partition function Z.

(ii) Using the bridge relation:

$$U = NkT^2 \frac{\partial \ln Z}{\partial T}$$

or otherwise, show that the internal energy U can be written as:

$$U = 2N\varepsilon \frac{1+\exp(-\varepsilon/kT)}{2+\exp(-\varepsilon/kT)},$$

where k is the Boltzmann constant.

(iii) Derive the limits of U as $T \to 0$ and as $T \to \infty$.

(iv) Sketch the variation of U with T.

(v) Without further differentiation sketch the graph of heat capacity C_V versus T.

[10]

c) The Helmholtz free energy F of a system is given by:

$$F = U - TS\ ,$$

where U is the total internal energy, T the thermodynamic temperature, and S the entropy.

(i) Using the first law of thermodynamics in the form:

$$dU = T\ dS - p\ dV\ ,$$

show that the entropy is related to the Helmholtz free energy by:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V\ ,$$

and that the pressure is related to the Helmholtz free energy by:

$$p = -\left(\frac{\partial F}{\partial V}\right)_T\ .$$

(ii) In the classical limit, the partition function for a Maxwell-Boltzmann gas can be written as:

$$Z = \frac{V}{V_0}\left(\frac{T}{T_0}\right)^{\frac{3}{2}},$$

where V_0 and T_0 are constants. Using the bridge relation:

$$F = -NkT\ln\left(Z\right),$$

find expressions for the entropy and for the pressure of a Maxwell-Boltzmann gas, as functions of volume and temperature.

[8]

d) A system consists of N fermions that can exist in states uniformly distributed in energy. The number of fermions $n(\varepsilon)d\varepsilon$ in the energy range ε to $\varepsilon+d\varepsilon$ is given by:

$$n(\varepsilon)\,d\varepsilon = \frac{g\,d\varepsilon}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1},$$

where $g\,d\varepsilon$ is the number of states in the energy range ε to $\varepsilon+d\varepsilon$, μ is the chemical potential, T is the thermodynamic temperature, and k is Boltzmann's constant.

(i) Show that the chemical potential is given, as a function of temperature, by:

$$\mu(T) = kT\ln\left[\exp\left(\frac{N}{gkT}\right)-1\right].$$

(ii) Show that:

$$\varepsilon_F = \frac{N}{g},$$

where the Fermi energy ε_F is defined as the value of the chemical potential $\mu(T)$ in the limit $T \to 0$. Explain the physical significance of the Fermi energy.

(iii) The Fermi temperature T_F is defined by $\varepsilon_F = kT_F$. Sketch a plot showing $n(\varepsilon)$, from $\varepsilon = 0$ to a value $\varepsilon > \varepsilon_F$, in three cases: in the limit $T \to 0$; for $0 < T << T_F$; and for $T > T_F$. Indicate clearly on the horizontal axis on your plot the point $\varepsilon = \varepsilon_F$.

[8]

[You are given that:

$$\int_0^\infty \frac{dx}{\exp(\beta x-\alpha)+1} = \frac{1}{\beta}\ln[1+\exp(\alpha)]. \qquad]$$

e) (i) Explain what is meant by Bose-Einstein condensation.

(ii) A system of N atoms of liquid helium-4 (He^4) contained in a volume V has a Bose temperature T_B given by:

$$T_B = \left(\frac{h^2}{2\pi mk}\right) \cdot \left(\frac{N}{2.612V}\right)^{\frac{2}{3}},$$

where h is Planck's constant, k is Boltzmann's constant, and m the mass of an atom.

Estimate T_B for liquid He^4, which has a molar volume of 27×10^{-6} m^3.

[6]

f) (i) State the nature of electric current carriers in normal conductors and superconductors.

[2]

(ii) Draw diagrams showing the constituents of an atom of He^3 and one of He^4. Using these, explain why one atom is a fermion and the other is a boson.

[5]

(iii) The critical field $B_C(T)$ in a superconductor at temperature T is related to that at T = 0, $B_C(0)$, by the relationship:

$$B_C(T) = B_C(0)\left[1-(T/T_C)^2\right]$$

where T_C is the critical temperature for B = 0. Sketch the relation $B_C(T)$ versus T and label the regions of normal conductivity and superconductivity.

[2]

(iv) Lead has T_C = 7.2 K and $B_C(0)$ = 0.08 T. Is lead at T = 6.0 K inside a field B = 0.04 T in a superconducting state?

[1]

2. Answer **<u>either</u>** 2(a) **<u>or</u>** 2(b).

a) A Maxwell-Boltzmann gas consists of N monatomic particles in a box with rigid walls. The box is in the shape of a cube with side length L. The wave function for a single particle can be written as:

$$\psi = \psi_0 \sin(k_x x)\sin(k_y y)\sin(k_z z),$$

where ψ_0 is a constant.

(i) Show that possible states have wave numbers k_x, k_y and k_z that satisfy:

$$k_x = \frac{\pi}{L} n_x, \qquad k_y = \frac{\pi}{L} n_y, \qquad k_z = \frac{\pi}{L} n_z,$$

where n_x, n_y and n_z are positive integers (1, 2, 3...) Sketch a plot showing the distribution of states in k-space, and hence show that the number of states with wave number $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ between k and $k+dk$ is $g(k)\,dk$, given by:

$$g(k)dk = \frac{V}{2\pi^2} k^2 dk,$$

where $V = L^3$ is the volume of the box.

[5]

(ii) Write down an expression relating the energy ε of a state to the wave vector k. Hence, show that the number of states with energy between ε and $\varepsilon+d\varepsilon$ is $g(\varepsilon)\,d\varepsilon$, given by:

$$g(\varepsilon)d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\varepsilon}\, d\varepsilon.$$

[4]

2. a) (continued)

The number of particles $n(\varepsilon)\, d\varepsilon$ with energy between ε and $\varepsilon+d\varepsilon$ is given by the Boltzmann distribution:

$$n(\varepsilon)\,d\varepsilon = \frac{N}{Z}\,g(\varepsilon)\,e^{-\frac{\varepsilon}{k_B T}}\,d\varepsilon\ ,$$

where Z is the partition function, and k_B is Boltzmann's constant.

(iii) Write an expression for Z in terms of the density of states $g(\varepsilon)$. Using this expression for Z, the Boltzmann distribution, and the above expression for the density of energy states $g(\varepsilon)$, show that the number of particles in the Maxwell-Boltzmann gas with energy between ε and $\varepsilon+d\varepsilon$ can be written:

$$n(\varepsilon)\,d\varepsilon = N\frac{2}{\sqrt{\pi}}\left(k_B T\right)^{-\frac{3}{2}}\sqrt{\varepsilon}\,e^{-\frac{\varepsilon}{k_B T}}\,d\varepsilon\ .$$

[4]

(iv) Using the above expression for $n(\varepsilon)\, d\varepsilon$, show that the total energy U of the gas is given by:

$$U = \frac{3}{2}Nk_B T\ .$$

[2]

(v) Write an expression for the energy ε of a particle in terms of its speed v. Hence, show that the number of particles in the gas with speed between v and $v+dv$ is given by the Maxwell-Boltzmann distribution:

$$n(v)dv = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)dv\ .$$

[4]

(vi) Sketch a graph of $n(v)$ versus v, and indicate the most probable speed v.

[3]

(vii) Find an expression for the mean square speed $\langle v^2 \rangle$, and show that your result is consistent with the above expression for the total energy of the gas.

[3]

[You are given that:

$$\int_0^\infty x^{\frac{1}{2}}e^{-x/a}dx = \frac{\sqrt{\pi}}{2}a^{\frac{3}{2}},\qquad \int_0^\infty x^{\frac{3}{2}}e^{-x/a}dx = \frac{3\sqrt{\pi}}{4}a^{\frac{5}{2}},\qquad \int_0^\infty x^4 e^{-x^2/b^2}dx = \frac{3\sqrt{\pi}}{8}b^5.\ ]$$

2. (continued)

b) Phonons propagate in a solid shaped as a cube with side length L.

(i) Given that the phonons cannot propagate outside the solid, show that possible states have wave numbers k_x, k_y and k_z that satisfy:

$$k_x = \frac{\pi}{L} n_x, \qquad k_y = \frac{\pi}{L} n_y, \qquad k_z = \frac{\pi}{L} n_z,$$

where n_x, n_y and n_z are positive integers (1, 2, 3...) Sketch a plot showing the distribution of states in k-space, and hence show that the number of states with wave number $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ between k and $k+dk$ is $g(k)\,dk$, given by:

$$g(k)dk = \frac{3V}{2\pi^2} k^2 dk,$$

where $V = L^3$ is the volume of the box.

[5]

In the Debye model, the relation between phonon frequency ν and the magnitude k of the wave vector is:

$$\nu = \frac{kc}{2\pi},$$

where c is the velocity of sound.

(ii) Show that the density of phonon states as a function of frequency ν can be written:

$$g(\nu)d\nu = \frac{12\pi V}{c^3} \nu^2 d\nu .$$

[3]

(iii) A solid containing N atoms supports $3N$ phonons. Show that for such a solid, the cut-off phonon frequency ν_D is given by:

$$\nu_D^3 = \frac{3c^3}{4\pi} \frac{N}{V} .$$

[3]

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2. b) (continued)

(iv) The phonon energy can be written:

$$U = \int_0^{\nu_D} \frac{12\pi V}{c^3}\nu^2 h\nu \cdot \frac{1}{\exp(h\nu/k_B T)-1}\cdot d\nu \, .$$

Explain the origin of the factors $h\nu$ and $\frac{1}{\exp(h\nu/k_B T)-1}$.

[2]

(v) Writing $y = h\nu/k_B T$, and $y_D = h\nu_D/k_B T$, show that:

$$U = \frac{12\pi V k_B^4 T^4}{c^3 h^3}\int_0^{y_D}\frac{y^3 dy}{e^y - 1}.$$

[2]

(vi) Show that, in the low temperature limit $T \to 0$, the total energy U tends to:

$$U \to \frac{4\pi^5 V k_B^4 T^4}{5c^3 h^3}.$$

Show also that, in the high temperature limit where $k_B T >> h\nu_D$, the total energy is given approximately by:

$$U \approx 3Nk_B T \, .$$

[6]

(vii) Derive expressions for the heat capacity C_V in the low temperature and high temperature limits. Sketch a graph of C_V versus T.

[4]

[You are given that:

$$\int_0^{\infty}\frac{y^3 dy}{e^y - 1} = \frac{\pi^4}{15}. \ ]$$

3. Answer **either** 3(a) **or** 3(b).

a) (i) Describe qualitatively the basic features of the theory of superconductivity.

[5]

(ii) Explain qualitatively what happens as the temperature of a superconductor rises above the critical temperature.

[2]

(iii) Explain qualitatively what happens as a magnetic field higher than the critical field is applied to a superconductor.

[2]

(iv) Describe the isotope effect in superconductivity. What does it prove?

[4]

(v) Describe the Meissner effect.

[5]

(vi) Give two main characteristics of high-T_C superconductor materials.

[2]

(vii) Is high-T_C superconductivity fully understood?

[1]

(viii) How does the magnetic flux quantum confirm the BCS theory? Mention one important application of magnetic flux quantization.

[4]

3. (continued)

b) (i) Draw the pressure versus temperature phase diagram for He^3 and label the different phases (assume no external magnetic field).

[6]

(ii) Does He^3 become superfluid? If yes, at what temperature? Mention three key features of He^3 behaviour in the relevant temperature range.

[5]

(iii) Describe nuclear adiabatic demagnetization cooling.

[10]

(iv) Describe briefly the two-fluid model for He^4 II and present the main experimental support for it.

[4]